In 1997 Julie Foudy, then a USA woman's national soccer team midfielder and former member of the 1966 USA Olympic championship team, currently a soccer announcer and commentator, went to Pakistan to check out reports that soccer balls were produced by child labor. As part of her report she noted that she learned that the soccer ball used at that time, called here the "traditional soccer ball" always had 32 panels, 12 pentagons and 20 hexagons that were to be sewn together by hand. *[Michael Lewis, Soccer Magazine, 1997]*

There is a simple explanation of Foudy's observation which I was then motivated to show my elementary mathematics classes, and which I give below. However the early Greek mathematicians knew about the the Platonic solids of which the icosahedron has 20 faces and 12 vertices while the dodecahedron has 12 faces, 20 vertices, both have 30 edges. The main thrust of this article is the connection between the platonic solids and paneling of soccer balls.

As part of my preparation of this article I stopped at my local soccer store to obtain a real traditional ball to check that the pentagons and hexagons where really regular, unlike in many inexpensive balls. I was surprised to find out that traditional balls were no longer sold. Now each vendor, each league and even each tournament has a different ball. Nico Colamussi, the clerk on duty at the *Soccer and Rugby Imports* in Branchville Connecticut, kindly tidied up the display of some of the many balls sold by his store so I could take a picture.

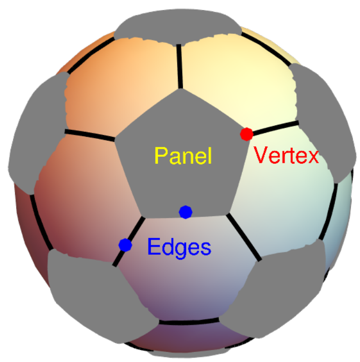
Nowadays basketballs and volleyballs also come in varying color balls but generally on the same panel base unlike soccer where the balls have different paneling. Still the traditional soccer ball is often used as a paneling as well as the dodecahedron. I did buy two balls for comparison, one with traditional paneling by Adidas and a dodecahedral paneled ball by Nike.

Motivated by this I will view polyhedral solids as polygonal panelings of a sphere, rather than polyhedral solids. It turns out that it is not only the topology that is important but that most of these balls, like the traditional one, together with the icosahedron and dodecahedron have the same rotational symmetries. In this paper a soccer ball will denote a polygonal paneling of the sphere with this group of rotational symmetries. The main tool here will be a fixed set of 62 points which are intersections of axes of rotational symmetry with the sphere. I will end with the construction of recent ball, the Puma official ball of the 2024 Copa America tournament and a replica 2024 Olympic ball by Adidas which clearly illustrates these points.

In a future article I will give details and algorithms on actually producing the graphics shown here as well as the polyhedral equivalents using computer algebra system `Mathematica`. My constructions in this article will avoid computer technicalities, but will mention some mathematical ideas including Euler's Polyhedral Theorem.

1. The traditional ball and Euler's Theorem.

A ball polygonal paneling is what is known mathematically as a *spherical polyhedron.* The 18th century mathematician Leonhard Euler had a theorem about these. It said $F-E+V=2$. Here F is the number of faces, or as we will say, *panels* . The *Edges* give the outline of the panels, in this case what makes a paneling polygonal is that they are portions of great circles on the ball. Each panel will have at least 3 edges. Generally in soccer ball construction they are the stitching between panels. Finally the *vertices* are end points of edges where they meet other edges. Each edge will have two vertices. A simple discussion of why Euler's theorem works is given in my website [barryhdayton.space/NEIUarchives/soccerBall.pdf] which gives my original classroom discussion of this. In my notation the formula above will be written $P-E+V=2$ where P is the number of panels, E the number of edges and V the number of vertices.



Since each panel of the traditional soccer ball is a pentagon or hexagon,we will let h denote the number of hexagons and p the number of pentagons. So $P = p+h$.

But each hexagon has 6 edges and each pentagon has 5, but each edge is an edge of two panels. So the number of edges is $\frac{5p+6h}{2}$.

We notice that each vertex occurs only once as a vertex of a pentagon so $V=5p$, on the other hand each vertex is a vertex of 2 hexagons so $V=6h/2=3$ h. Combining these 2 equations gives $5p=3h$ or $h=\frac{5p}{3}$ so we can write $P-E+V$ entirely in terms of p. We get the somewhat complicated equation

$$(p+(5\ p/3))-(6\times(5\ p/3)+5\ p)/2+5\ p = 2$$

Doing the algebra the common denominator on the left is 6 and we get simply

$$p/6 = 2$$

or $p = 12$. But using our formula $h=\frac{5p}{3}$ above we get also $h=20$. So we have 12 pentagons and 20 hexagons for the 32 panels that Julie Foudy counted. There are 30 edges from the formula above or Euler's formula $12-30+20=2$. These numbers 12, 20 and 30 will play an important role in this paper.

2. Symmetry

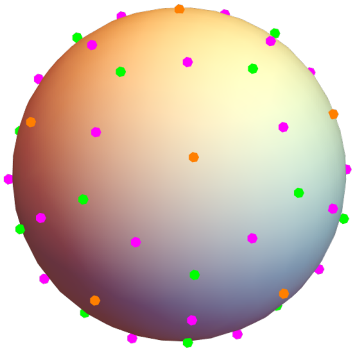
There are many panelings of the sphere and all obey Euler's Polyhedral Theorem. One additional property shard by most soccer ball panelings is that the panelings are symmetric. That is if placed at rest they always look about the same. Some modern balls do have writing or graphics on them but most have symmetric paneling so they will roll smoothly. The surprise is that most of these panelings have the same rotational symmetries that Plato discovered on the icosahedron and dodecahedron.

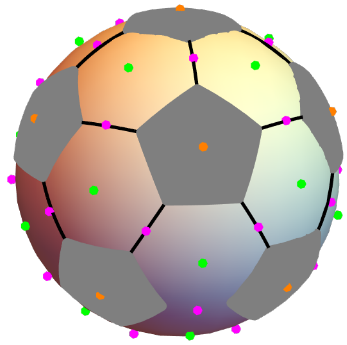
On the picture in the introduction we see that the pentagon in the middle is regular, all sides are the same length. This forces it to be symmetric about its centroid, in this case by a 1/5th turn, that is a 72° turn either clockwise or counter clockwise. Note that if we rotate the ball in the axis through the centroid and center of the ball then the pentagons connected to this ball by an edge rotate to an adjoining pentagon as do the the hexagons next to this pentagon. In both cases there are 5 of them.

It is not so easy to see that the hexagons are regular and symmetric due to the coloring on the hexagons. In fact, in many inexpensive traditional balls the hexagons are not regular. Below is a picture an Adidas ball with traditional paneling but irregular marking where it is easier to see that the hexagon of the front center is seen to be quite regular. A $1/6^{th}$ turn, 60°, takes the hexagon to itself, but not the marking. There are only 3 pentagons surrounding this hexagon and a symmetric rotation of the entire ball must take these pentagons to pentagons. So the marking on this hexagon shows that we only have $1/3$ turn symmetry, 120°, of the traditional ball about the centroid of the hexagon.

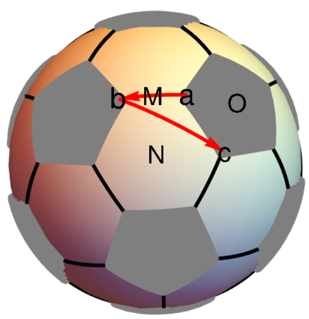
It is harder to see but these traditional balls also have a half turn, 180°, about the midpoint of the edges between two pentagons. On the Adidas ball the markings point to the axes of this rotations, in our earlier picture the blue dot on the edge going from the middle panel to the lower left panel shows where the axis is. These same symmetries are found on every hexagon or pentagon and edge between two pentagons but are the only possible rotations that preserve the paneling. It can be shown that there are 60 such rotations noting that in addition to the two 72°rotations of the pentagons there are also two 144° rotations and we count not rotating the ball at all as one of the rotations. There are only the 3 rotations of the hexagon, counting the non-rotation, and two half turns about each half turn axis.

Following are graphics showing the centers of the possible rotation symmetries and the centers super-imposed on the traditional ball . The orange dots are centers of 5-fold rotation, the green dots are the 3-fold centers and the magenta dots are for the half turns. This set of 62 points is unique only up to rotation of the entire ball but the relative positions are unique. This set will be used throughout this paper.





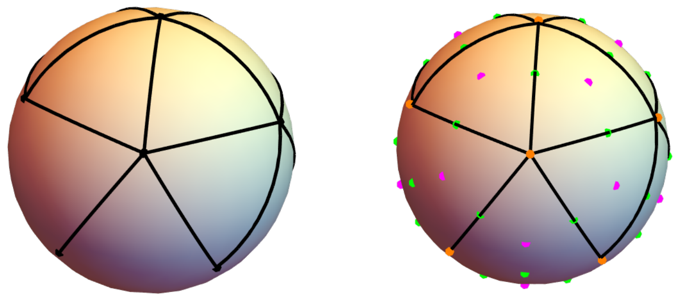
These rotations can be combined . So suppose, in the graphic below , that we first do a half turn in the axes with end “M”, then the pentagon vertex marked “a” gets sent to the point marked “b”. Now we rotate 1/3 turn clockwise on the axes marked N which ends in the center of a hexagon. That moves the veertex “b” to the vertex “c” of that hexagon.



But "c" is on the original pentagon with centroid “O”and could have been rotated there just by a 1/5 turn counterclockwise about “O”. In fact it can be shown that the combined rotation of a half turn about M followed by a 1/3 turn clockwise around N is just the rotation given by rotation of 1/5 turn counterclockwise about “O” for all points on the ball. This might seem bizarre to someone who knows plane transformation geometry, but it works on the surface of a ball and explains why pentagons are such common figures in paneling a soccer ball. More generally we will see later that actually it is figures with a symmetry preserved by a 1/5 turn that are important.

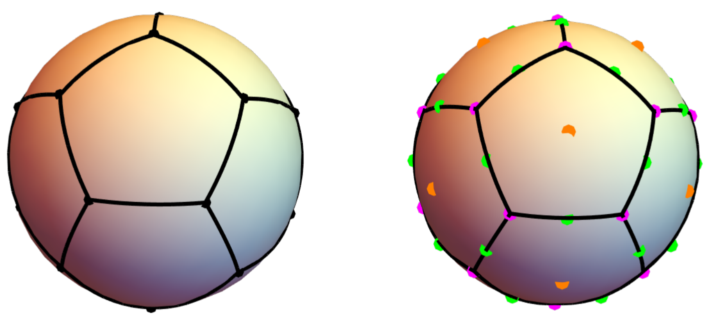
3. The icosahedron and dodecahedron balls

The icosahedron and dodecahedron balls can be constructed directly from this set of rotation centers. Since the icosahedron is known to have 12 vertices we can use the 1/5 turn axes. Here are graphics of the icosahedron ball without and with the rotation centers.



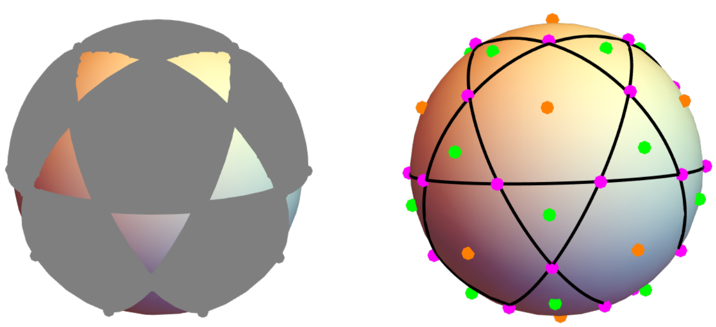
This might make a nice soccer ball with a small number of panels, perhaps with graphics or text in the panels and different color panels. However, the 5 panels coming together at a vertex could be a sewing problem.

For the dodecahedron there are 20 vertices so we can use the 1/3 turn axes as vertices.



This is actuated by many Nike balls such as the one in the picture above of two balls.

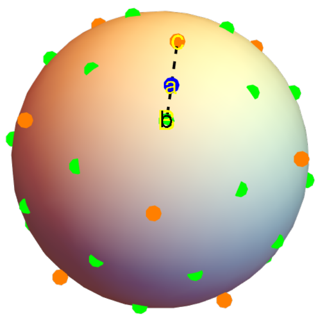
With these two well known geometrical shapes we add another ball with a less well known shape, the icosidodecahedron . Here there are 12 pentagons, 20 triangles but 30 vertices, so we can use the 1/2 turn axes as vertexes. It is a bit more complicated ,so I color the panels on the left to make it easier to understand. The right hand picture again shows the axes points.



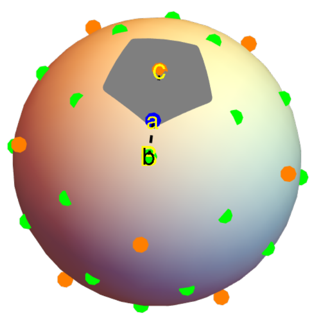
3. Constructing the traditional ball and variations.

Constructing a perfect traditional soccer ball, one with regular hexagons, is difficult, especially starting from a pentagon. Later we will show how to construct a perfect ball from a hexagon. However our general method has an added advantage. We can construct a continuum of paneling with limiting figures the icosahedron and icosidodecahedron and the traditional soccer ball somewhere in the middle.

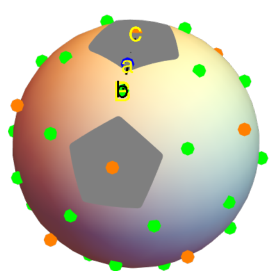
Step 1: Pick a point on a circular arc strictly between a 1/5 turn, marked c, and a nearby 1/2 turn axis, it b. The point we pick will be called **a**.



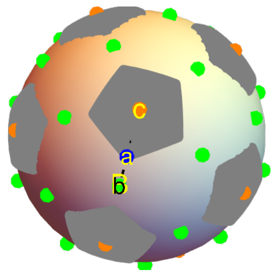
Step 2: Rotate a 1/5, 2/5, 3/5 and 4/5 turn about c to get a pentagon, We will call this pentagon1. Note the vertex at a.



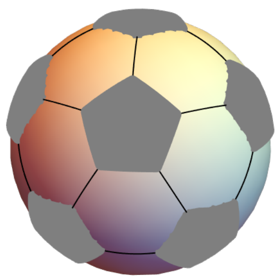
Step 3 : Now execute 1/2 turn about b of pentagon1 to get pentagon2. Note that the centroid of pentagon2 is another 1/5 turn axis point and b is the midpoint between that axis point and c. Also note that vertices of these two pentagons are facing each other.



Step 4 : Rotate pentagon2 around c by 1/5, 2/5, 3/5, 4/5 turns to get 4 new pentagons.



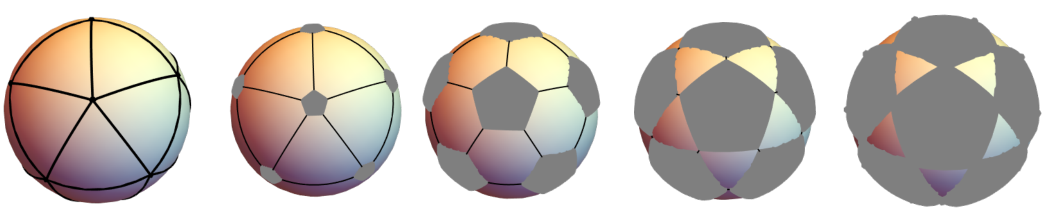
Step 5: We next add in the stitching, that is edges between nearby pentagons.



When we measure the lengths of the edges of the hexagons we find that the edges that are on one pentagon compared to the edges that go between pentagons are 12% shorter. So we needed to start with a larger pentagon1 to get a perfect traditional soccer ball. Later we will use a different method to get a perfect ball.

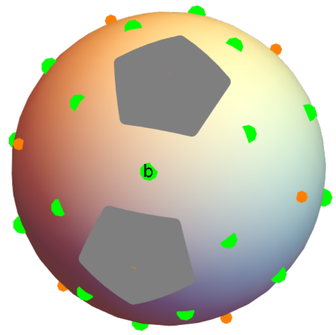
4. A continuum of imperfect balls and other like balls.

If, in Step 1 of the construction above we adjust point a we can get other panelings. Here is a continuum

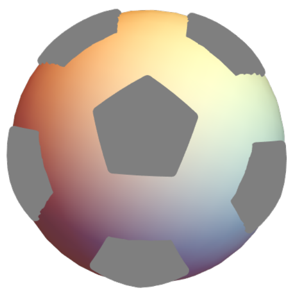


starting from the icosahedron on the, our imperfect ball of section 3 in the middle and the icosidodeca - hedron on the right. Note the 5 triangular panels showing on both the right and left. In the middle these become hexagons.

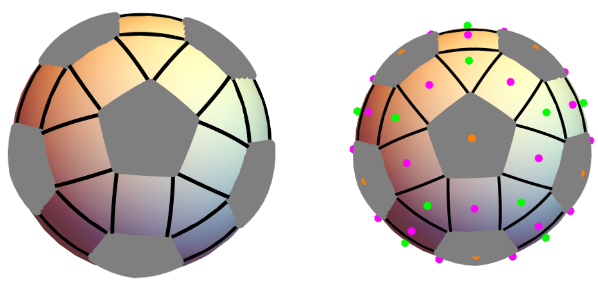
The important thing about pentagons is that they have 1/5 turn rotational symmetry. We could replace our pentagon above by a differently oriented pentagon or a different shape as long as this symmetry holds. The *Copa America* ball below will be a good example. For now we turn our pentagon upside down with surprising results. Step 3 in our earlier construction now gives



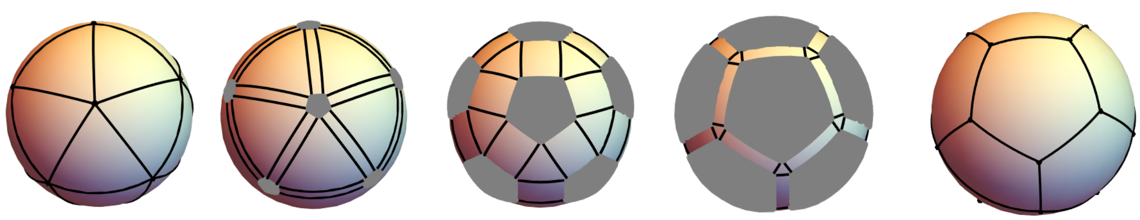
where the vertices are pointing away from point b . Continuing on with step 4 gives the picture



But now the stitching is different, we get rectangles and triangles instead of hexagons. In fact each panel corresponds to one axis point at its centroid for 62 panels.



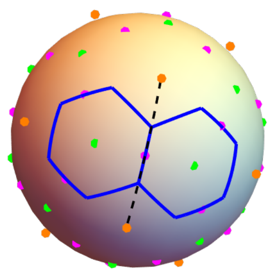
This gives us a continuum from the icosahedron to the dodecahedron.



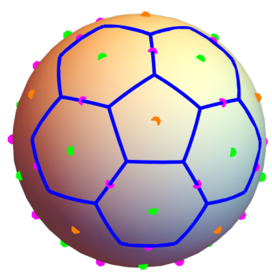
5. The perfect traditional ball

In the last two sections we started with a pentagon of varying sizes and orientations and came up with a paneling. However we had to estimate the proper size pentagon and did not quite get the perfect ball with regular hexagons as well as pentagons. The problem is that the symmetry we are requiring on the hexagon was only 1/3 turn symmetry but to get a regular pentagon we need 1/6 turn symmetry. So the trick is to start with a regular hexagon. Since we are still requiring the same symmetry the pentagon still has the 1/5 turn symmetry and so our construction will require it to be regular also.

Unfortunately drawing a regular hexagon on a ball with the same spherical lengths of the sides is a much harder problem than in the plane case. Specifically the Pythagorean theorem fails on the sphere. So the only method I know to give the right answer requires algebra and the solution of non-linear equations. So I will leave this for my more technical paper. The answer is that the common edge length is precisely, not approximately or exactly, 0.406337892071491×(radius of ball) . This gives the following two hexagons where the green dots which are centroids of these hexagons are our 1/3 turn axes points. The line between two orange 1/5 turn axes coincides with common side and a magenta half turn axis is the midpoint of the common side, in fact three of the sides of each have a 1/2 turn axis point as a midpoint.

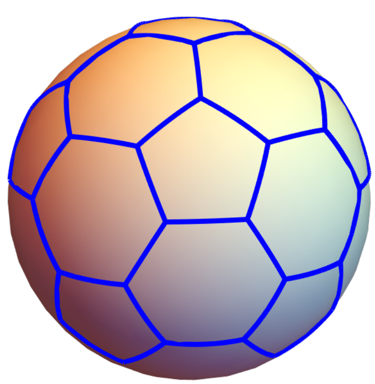


If we propagate these hexagons along the half turn axis midpoints we get the picture:



Notice here we have 5 regular hexagons bounding a pentagon, something you won't see in the plane. Also important is that 1/5 turn symmetry was not used to get this graphic.

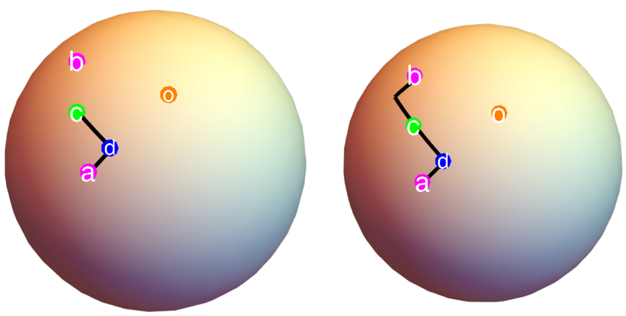
So here is our perfect traditional soccer ball, shown without coloring so as not to obscure the edges.



6. The Copa America 2024 Ball

As mentioned in the Introduction, Puma designed and manufactured a ball for the recently completed 2024 *Copa America* tournament. This is a good example of our methods as we can easily describe the paneling.

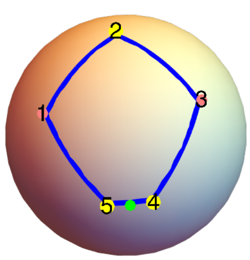
Start by picking a 1/5 turn axis point for the center of a panel, somewhere near the point o above. We see two obvious 1/3 axis points near a,b to use. The edge stitching between these points is not straight but does seem to have half turn symmetry about a point near c so that point should be the 1/2 turn axis between a,b. Thus we need to set the point d where this edge turns, unfortunately d is not an axis point and is somewhat arbitrary. Then rotating about c will give the whole edge. Once we are satisfied we can rotate 5 times around the 1/5 turn axis near o which will be the centroid of the panel.

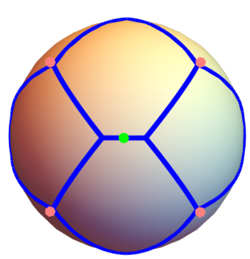


Once we are satisfied we can rotate 5 times around the 1/5 turn axis at o, the centroid of the panel. We apply our 1/5 turn about o four times to the segment between a and b to get our panel outlined in green on the left. Finally we populate the this panel as we did with steps 3,4 to get approximation of the paneling of the Copa American Ball on the right.

7. The Adidas Olympic Replica Ball

The actual ball used in the 2024 Olympics,on the left, is complicated and the panels are not polygonal, so they do not fit this article. However Adidas has marketed several less expensive replica balls such as the one on the right. It is this ball that I will discuss here. There are 12 polygonal panels one of which is shown above, although is is a bit hard to see that the top of this pentagon is in the white circular region. But if you look at the identical white circular region at the bottom of the picture you will see the white tops of two identical horizontally located pentagons. Unfortunately these pentagons do not have a rotational symmetry so we do not have a dodecahedral ball. But Adidas clearly identifies two 1/3 symmetries at the sides of the panel with solid colored blue regions, the left one light and the right darker. Also the center of the white circle is clearly an axis point of a 1/2 turn. If we place these points on our ball of axis points in section 2 a similar relative position of 1/3 and 1/2 turns. If we assume these point then we can find the two bottom points,marked 4,5 below left, and the top point, marked 2, where the top sides and bottom sides intersect. These are not axis points and as with our perfect traditional ball require some heavy math to get precisely. The green point at the bottom is one of our half turn points and the side points, marked 1 and 3, are our 1/3 turns.





Above right is our completed paneling of this ball, the known half turns are marked in green while the 1/3 turns are in pink. We do not have 1/5 turn symmetry for this ball so our full group of rotational symmetries is considerably smaller than for our traditional ball.