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# ON THE NATURALITY OF PIC, $SK_0$ AND $SK_1$

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ABSTRACT. Several facts about $SK_0$ and $SK_1$ are presented, both for commutative rings and schemes. If A is the homogeneous coordinate ring of a projective variety over a field k, then Pic(A), $SK_0(A)$ and $SK_1(A)$ are naturally modules over the ring W(k) of Witt vectors over k. If A is any commutative ring, NPic(A), $NSK_0(A)$ and $NSK_1(A)$ are naturally modules over W(A). The K-theory transfer map, defined when B is an A-algebra which is a finite projective A-module, sends $SK_0(B)$ to $SK_0(A)$ and $SK_1(B)$ to $SK_1(A)$.

## 0. INTRODUCTION

The main goal of this paper is to prove that if A is the homogeneous coordinate ring of a projective variety over a field k, then

$$0 \longrightarrow SK_0(A) \longrightarrow \tilde{K}_0(A) \xrightarrow{\det} \mathrm{Pic}(A) \longrightarrow 0$$

is a short exact sequence of modules over the ring W(k) of Witt vectors of k. Here $\tilde{K}_0(A)$ is the kernel of the rank function from $K_0(A)$ to the ring $H^0(A)$ of all continuous functions $\mathrm{spec}(A) \longrightarrow \mathbb{Z}$, and $SK_0(A)$ is the kernel of the map

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$$\det\colon \tilde{K}_0(A) \longrightarrow \mathrm{Pic}(A).$$

(See [Bass, IX.3], where $\tilde{K}_0(X)$ is called $Rk_0$.) Since A is graded, $\tilde{K}_0(A)$ has a natural W(k)-module structure by [Wmod], so the main content of this result is that the map $\det\colon \tilde{K}_0(A) \longrightarrow \mathrm{Pic}(A)$ endows Pic(A) with the structure of a module over the ring W(k).

In order to prove this result, we needed to use the following fact: if an A-algebra B is a finitely generated projective A-module, then the transfer map $K_0(B) \longrightarrow K_0(A)$ takes $SK_0(B)$ to $SK_0(A)$. To our surprise, we could not locate this result in the literature. We could also not locate the well-known fact that projective modules of rank n and determinant 1 may be obtained by patching free modules by matrices in $SL_n$.

Even the fact that $SK_0(A)$ is an ideal of the ring $K_0(A)$ was hard to locate, although it is easy to prove using the splitting principle. Another proof is to observe that $SK_0(A)$ is the subgroup $F^2(A)$ in Grothendieck's $\gamma$-filtration

$$\cdots \subset F^2(A) \subset F^1(A) = \tilde{K}_0(A) \subset F^0 = K_0(A).$$

(See theorem 5.3.2 of [SGA6, Exposé X] or [FL, p. 126]). Since the $F^i(A)$ are ideals in the ring $K_0(A)$, it follows that $SK_0(A)$ is an ideal.

We have therefore decided to err on the side of completeness, and have organised our paper as follows. In the first three sections we consider the transfer map. Let B be an A-algebra which is a finitely generated projective A-module, so that the transfer map

$\pi_*: K_i(B) \longrightarrow K_i(A)$ is defined. In section 1 we show that $\pi_*$ takes $\tilde{K}_0(B)$ to $\tilde{K}_0(A)$; in section 2, we show that $\pi_*$ takes $SK_1(B)$ to $SK_1(A)$. In section 3, we show that $\pi_*$ takes $SK_0(B)$ to $SK_0(A)$ using the above result about $SK_1$ and a patching interpretation of $SK_0$ we have relegated to the appendix.

All of the above results apply more generally to finite scheme maps $\pi: X \longrightarrow Y$ such that $\pi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module. For such maps, $\pi_*$ is an exact functor from locally free $\mathcal{O}_X$-modules to locally free $\mathcal{O}_Y$-modules, so that the transfer map $\pi_*: K_i(X) \longrightarrow K_i(Y)$ is defined. In this paper, we have focussed as much as possible on the ring-theoretic results, because they are less 'hi-tech' than their scheme-theoretic analogues.

One interesting scheme-theoretic implication of these results is a simple Riemann-Roch type theorem (in the formalism of [FL]): for every finite map $\pi: X \longrightarrow Y$ of schemes with $\pi_*\mathcal{O}_X$ locally free, the diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{(\text{rank},\det)} & H^0(X,\mathbb{Z}) \oplus \mathrm{Pic}(X) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ K_0(Y) & \xrightarrow{(\text{rank},\det)} & H^0(Y,\mathbb{Z}) \oplus \mathrm{Pic}(Y) \end{array}$$

commutes. (See (3.4).)

In §4 and §5 we prove our module structure results, which clarify the results in [Swan, §8]. Our general result is that if $A = R \oplus A_1 \oplus \dots$ is a graded commutative ring, then $\mathrm{Pic}(A,A_+)$ is a $W(R)$-module, and if $S \subset R$ is a multiplicatively closed set, then $\mathrm{Pic}(S^{-1}A,S^{-1}A_+)$ is $W(S^{-1}R) \otimes \mathrm{Pic}(A,A_+)$.

In §6 we extend the above results from the subgroup $SK_0$ of $K_0$ to the subgroups $FL^nK_0$ of $K_0$ defined by Fulton and Lang in [FL, p.120]. We would like to thank C. Pedrini for pointing out that our methods could be applied to the groups in the Fulton-Lang filtration.

Finally, we have included an appendix on patching vector bundles, because we need some patching results we cannot find in the literature. For example, if P is a vector bundle on X with $\det(P) \in \mathrm{Pic}(X)$ trivial, then we can obtain P by patching free modules on an open cover $\{U\}$ of X via matrices in the $SL_n(U \cap V)$.

## 1. TRANSFER AND $\tilde{K}_0$

When A is a commutative ring, $K_0(A)$ is naturally the direct sum of $\tilde{K}_0(A)$ and $H^0(A)$. When B is a commutative A-algebra which is a finite projective A-module, the transfer map $\pi_*: K_0(B) \longrightarrow K_0(A)$ need not send $H^0(B)$ to $H^0(A)$ because $[B] \in K_0(A)$ need not belong to $H^0(A)$. However, it always sends $\tilde{K}_0(B)$ to $\tilde{K}_0(A)$:

1.1. <u>Proposition</u>: If B is a commutative A-algebra which is a finite projective A-module, then the transfer map $\pi_*: K_0(B) \longrightarrow K_0(A)$ sends $\tilde{K}_0(B)$ to $\tilde{K}_0(A)$, and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{K}_0(B) & \longrightarrow & K_0(B) & \xrightarrow{\text{rank}} & H^0(B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle N_{B/A}} & & \\ 0 & \longrightarrow & \tilde{K}_0(A) & \longrightarrow & K_0(A) & \xrightarrow{\text{rank}} & H^0(A) & \longrightarrow & 0, \end{array}$$

where $N_{B/A}$ is the composite $H^0(B) \subset K_0(B) \xrightarrow[\pi_*]{} K_0(A) \xrightarrow{\text{rank}} H^0(A)$.

Proof: It is enough to show that for every $\xi \in \tilde{K}_0(B)$ the function

$$\mathrm{rank}(\pi_*\xi): \mathrm{spec}(A) \longrightarrow \mathbb{Z}$$

is zero at every prime ideal p of A. The rank of $\pi_*\xi$ at p is the value of $(\pi_*\xi) \otimes_A A_p$ in $K_0(A_p) \cong \mathbb{Z}$. Since $\pi_*$ is natural with respect to localization, $(\pi_*\xi) \otimes_A A_p = (\pi_p)_*(\xi \otimes_A A_p)$, where $(\pi_p)_*: K_0(B \otimes_A A_p) \longrightarrow K_0(A_p)$. On the other hand, $\xi \otimes_A A_p = 0$ because $\tilde{K}_0(B \otimes_A A_p)$ is zero, $B \otimes_A A_p$ being a semilocal ring. Hence $\mathrm{rank}(\pi_*\xi) = 0$ at every p. □

1.2. Remark: ([Bass, p.451]). The hypothesis that B be projective may be weakened to assume that $B \in H(A)$. That is, the A-module B has a finite resolution by finite projective A-modules.

Since the proof of (1.1) is scheme-theoretic, it also proves the analogous result for schemes, which we now formulate. Let $\pi: X \longrightarrow Y$ be a finite map of schemes such that $\pi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module. Then $\pi$ is locally $\mathrm{spec}(A) \longrightarrow \mathrm{spec}(B)$, where B is a finite projective A-module, and the transfer map $\pi_*: K_i(X) \longrightarrow K_i(Y)$ is defined.

1.3. Proposition. If $\pi: X \longrightarrow Y$ is a finite map of schemes such that $\pi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module, then $\pi_*$ sends $\tilde{K}_0(X)$ to $\tilde{K}_0(Y)$, and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{K}_0(X) & \longrightarrow & K_0(X) & \longrightarrow & H^0(X,\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* & & \\ 0 & \longrightarrow & \tilde{K}_0(Y) & \longrightarrow & K_0(Y) & \longrightarrow & H^0(Y,\mathbb{Z}) & \longrightarrow & 0. \end{array}$$

1.4. Remark: There is also a transfer map $\pi_*: K_0(X) \longrightarrow K_0(Y)$ defined for proper maps $\pi: X \longrightarrow Y$ of finite Tor-dimension [SGA6]. These will not usually send $\tilde{K}_0(X)$ to $\tilde{K}_0(Y)$. For example, let $k$ be a field and set $Y = \text{spec}(k)$, so that $\tilde{K}_0(Y) = 0$ and $K_0(Y) \cong \mathbb{Z}$ on generator $[k]$. If $X = \mathbb{P}^1_k$, then $\tilde{K}_0(X) \cong \mathbb{Z}$ on the class of $\xi = [\mathcal{O}_X] - [\mathcal{O}_X(-1)]$, but $\pi_*(\xi) = [k]$, which has rank 1. Similarly, if $X = \mathbb{P}^2_k$, then $SK_0(X) \cong \mathbb{Z}$, and the transfer $\pi_*: K_0(X) \longrightarrow K_0(k)$ sends $SK_0(X)$ isomorphically onto $K_0(k)$. In this case $\pi_*$ does not even send $SK_0(X)$ to $\tilde{K}_0(k)$.

## 2. TRANSFER AND $SK_1$

When $A$ is a commutative ring, $K_1(A) = GL(A)/E(A)$ is the direct sum of $A^*$, the units of $A$, and the group $SK_1(A) = SL(A)/E(A)$. [Bass, V.2]. When $B$ is an A-algebra which is finitely generated and projective as an A-module, then one can define both the norm homomorphism $N_{B/A}: B^* \longrightarrow A^*$ and the transfer homomorphism $\pi_*: K_1(B) \longrightarrow K_1(A)$.

The transfer homomorphism may be defined as follows [Milnor, p.138]. Embed $B$ in some $A^d$ as a direct summand. This gives an embedding of groups for each $n$:

$$GL_n(B) \longrightarrow Aut_A(B^n) \longrightarrow Aut_A((A^d)^n) = GL_{nd}(A).$$

The transfer map is obtained by abelianizing and taking the direct limit as $n \longrightarrow \infty$. The norm map may be defined by the formula $N_{B/A}(b) = \det(\pi_* b)$ for $b \in B^*$. (See [Milnor, 14.2].) The following simple example shows that $\pi_*$ does not always send the subgroup $B^*$ of $K_1(B)$ to the subgroup $A^*$ of $K_1(A)$.

2.1. Example: Let $A = \mathbb{R}[x,y]/(x^2+y^2 = 1)$ and let $B = A \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[t,t^{-1}]$ where $t = x - iy$. Relative to the basis $(1,i)$ of $B$ as an A-module, $t$ has the matrix

$$\pi_*(t) = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \in SL_2(A).$$

In fact, this matrix represents the non-trivial element of $SK_1(A) \cong \mathbb{Z}/2$ by [Milnor, 13.5] showing that $\pi_*(B^*)$ is not contained in $A^*$.

2.2. Theorem: If $B$ is a commutative A-algebra which is a finitely generated projective A-module, then the transfer homomorphism $\pi_*$ sends $SK_1(B)$ to $SK_1(A)$, and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SK_1(B) & \longrightarrow & K_1(B) & \xrightarrow{\det} & B^* & \longrightarrow & 0 \\ & & \downarrow \pi_* & & \downarrow \pi_* & & \downarrow N_{B/A} & & \\ 0 & \longrightarrow & SK_1(A) & \longrightarrow & K_1(A) & \xrightarrow{\det} & A^* & \longrightarrow & 0. \end{array}$$

Proof: It is enough to see that $N_{B/A}(\det_B g) = \det_A(\pi_* g)$ for every $g \in K_1(B)$. If $B$ is semilocal, so that $K_1(B) \cong B^*$, this follows from the formula for $N_{B/A}$. In general, suppose given $g \in K_1(B)$ and consider the ratio

$$u = N_{B/A}(\det_B g)/\det_A(\pi_* g) \in A^*.$$

For each maximal ideal $m$ of $A$, $B_m$ is a finite projective $A_m$-module, and the determinant, norm and transfer maps are natural with respect to this base change. Consequently, if $g_m \in K_1(B_m)$ denotes the image of $g$ then the image of $u$ in $A_m^*$ is

$$u_m = N_{B_m/A_m}(\det_{B_m}(g_m))/\det_{A_m}(\pi_{m*}g_m).$$

Because $B_m$ is semilocal, $u_m = 1$. Hence $\mathrm{ann}_A(u-1)$ is not contained in any maximal ideal of $A$, i.e., $u = 1$. □

2.3. Corollary: If $B$ is a direct sum of $A^d$, then the map $GL_n(B) \longrightarrow GL_{nd}(A)$ sends $SL_n(B)$ to $SL_{nd}(A)$.

2.4. Remark: More generally, whenever $\pi: A \longrightarrow B$ is such that $B \in H(A)$, i.e., the A-module $B$ has a finite resolution

$$0 \longrightarrow P_n \longrightarrow \ . \ . \ . \longrightarrow P_0 \longrightarrow B \longrightarrow 0$$

with the $P_i$ finitely generated projective A-modules, then the transfer map $\pi_*: K_1(B) \longrightarrow K_1(A)$ is defined [Bass, p. 451]. If we define $N_{B/A}: B^* \longrightarrow A^*$ to be $N_{B/A}(b) = \det(\pi_* b)$, then the proof of 2.2 goes through to show that $\pi_*$ takes $SK_1(B)$ to $SK_1(A)$.

It should not be surprising that Theorem 2.2 generalizes to schemes, since the proof uses local rings. The analogue for a scheme $X$ of the units in a ring are the global units, i.e., the group $H^0(X, \mathcal{O}_X^*)$. Since $\mathcal{O}_X^*$ is the sheafification of the presheaf $U \mapsto K_1(U)$, there is a natural map

$$\det: K_1(X) \longrightarrow H^0(X; \mathcal{O}_X^*).$$

If $SK_1(X)$ denotes the kernel of det, it is easy to see that

$$K_1(X) \cong H^0(X, \mathcal{O}_X^*) \oplus SK_1(X).$$

2.5. Theorem: Let $\pi: X \longrightarrow Y$ be a finite map of schemes such that $\pi_* \mathcal{O}_X$ is locally free. Then $\pi_*: K_1(X) \longrightarrow K_1(Y)$ sends $SK_1(X)$ to $SK_1(Y)$, and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SK_1(X) & \longrightarrow & K_1(X) & \xrightarrow{\det} & H^0(X,\mathcal{O}_X^*) & \longrightarrow & 0 \\ & & \downarrow \pi_* & & \downarrow \pi_* & & \downarrow & & \\ 0 & \longrightarrow & SK_1(Y) & \longrightarrow & K_1(Y) & \xrightarrow{\det} & H^0(Y,\mathcal{O}_Y^*) & \longrightarrow & 0. \end{array}$$

Proof: For each point $y \in Y$, the semilocal ring $\mathcal{O}_{X,y}$ is finite and projective as an $\mathcal{O}_{Y,y}$-module, so the proof of 2.2 goes through.

## 3. TRANSFER AND $SK_0$

In this section we prove the following result. Let B be a finite A-algebra which is projective as an A-module. Then $\pi_*$ sends $SK_0(B)$ to $SK_0(A)$, and the induced map from Pic(B) to Pic(A) sends L to $\det_A(L)/\det_A(B)$. When cloaked in scheme-theoretic guise, the result is as follows:

3.1. Theorem: Let $\pi: X \longrightarrow Y$ be a finite map of schemes such that $\pi_* \mathcal{O}_X$ is locally free. Then $\pi_*: K_0(X) \longrightarrow K_0(Y)$ sends $SK_0(X) \longrightarrow SK_0(Y)$, and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SK_0(X) & \longrightarrow & \tilde{K}_0(X) & \xrightarrow{\det} & \mathrm{Pic}(X) & \longrightarrow & 0 \\ & & \downarrow \pi_* & & \downarrow \pi_* & & \downarrow & & \\ 0 & \longrightarrow & SK_0(Y) & \longrightarrow & K_0(Y) & \xrightarrow{\det} & \mathrm{Pic}(Y) & \longrightarrow & 0. \end{array}$$

where $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$ sends a line bundle L on X to $\det_Y(L) \otimes \det_Y(\mathcal{O}_X)^{-1}$.

For expositional reasons, we first consider the case in which $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$ and $B$ is a free A-module of rank $d$. In this case the result looks like this:

3.2. <u>Corollary</u>: Let $\pi: A \longrightarrow B$ be a map of commutative rings such that $B \cong A^d$ as an A-module. Then $\pi_*: K_0(B) \longrightarrow K_0(A)$ sends $SK_0(B)$ to $SK_0(A)$, and the induce map $\mathrm{Pic}(B) \longrightarrow \mathrm{Pic}(A)$ sends $L$ to $\det_A(L) = \Lambda^d L$.

<u>Proof of 3.2</u>: Every element of $SK_0(B)$ can be written as $\xi = [P] - [B^n]$ for some rank $n$ projective B-module $P$ satisfying $\det(P) = B$. Since $\pi_*(\xi) = [P] - [A^{nd}]$, and $\pi_*(\xi) \in \tilde{K}_0(A)$ by 1.1, we only have to show that $\Lambda^{nd}P \cong A$. By A.3 there is a covering $\mathcal{U} = \{\mathrm{spec}(A[s^{-1}])\}$ of $\mathrm{spec}(A)$ so that the B-module $P$ is obtained by patching the modules $\{B^n[s^{-1}]\}$ via matrices $g_{st} \in SL_n(B[s^{-1}, t^{-1}])$. Embedding $SL_n(B)$ in $SL_{nd}(A)$ via 2.3, we see that the A-module $P$ is obtained by patching $\{A[s^{-1}]^{nd}\}$ via matrices in $SL_{nd}(A[s^{-1}, t^{-1}])$. As $\det(P)$ is obtained by patching $\{A[s^{-1}]\}$ via the determinants of these matrices, this implies that $\Lambda^{nd}P \cong A$, as desired. □

3.3. <u>Remark</u>: The above proof may be modified to prove 3.1 in the general affine case, i.e., when $B$ is a projective A-module. However, we cannot be as naïve about patching. The transfer $SL_n(B) \longrightarrow SL_{nd}(A)$ of 2.3 sends patching data for the B-module $P$ to patching data for an A-module of the form $P \oplus Q$, and sends patching data for $B^n$ to patching data for the A-module $B^n \oplus Q$. Since

$\det(P \oplus Q) = \det(B^n \oplus Q) = A$, we have

$$\pi_*(\xi) = [P] - [B^n] = [P \oplus Q] - [B^n \oplus Q] \in SK_0(A).$$

Such a proof will not work in the scheme case, however, because in general the vector bundle $\pi_*\mathcal{O}_X$ on Y cannot be embedded as a summand of a free $\mathcal{O}_Y$-module. Therefore, we leave the details of this remark to the reader.

Proof of 3.1: Every element of $SK_0(X)$ can be written as $\xi = [P] - [P']$, where $\mathrm{rank}(P) = \mathrm{rank}(P')$ and $\det(P) = \det(P')$. Adding line bundles to P and P', we can assume that $\det(P)$ is trivial. Replacing X by a component of X if necessary, we may assume that P and P' have constant rank n on X. As Y is a disjoint union of components on which $\mathrm{rank}_Y(\pi_*\mathcal{O}_X)$ is constant, we can restrict to such a component to assume that $\pi_*\mathcal{O}_X$ has constant rank d on Y. Choose an open cover $\mathcal{U}$ of Y so that: (1) the vector bundle $\pi_*\mathcal{O}_X$ on Y is obtained by patching free modules on the U in $\mathcal{U}$ via matrices $\beta_{UV} \in GL_d(U \cap V)$; (2) the vector bundles P and P' are obtained by patching free modules on the $\pi^{-1}(U)$ via respective matrices

$$g_{UV},\ g'_{UV} \in SL_n(\mathcal{O}_X|\pi^{-1}(U \cap V)).$$

Such a cover exists by A.3. Our task is to analyse the vector bundles $\pi_*P$ and $\pi_*P'$ on Y in terms of this data.

On each U, the trivializations of P and $\pi_*\mathcal{O}_X$ yield an isomorphism $\pi_*P|U \cong \mathcal{O}_U^{nd}$. On $U \cap V$, the two trivializations of $\pi_*\mathcal{O}_X$ yield two embeddings

$$\iota_U,\ \iota_V\colon SL_n(\mathcal{O}_X|\pi^{-1}(U\cap V)) \longrightarrow SL_{nd}(\mathcal{O}_{U\cap V}).$$

which differ by conjugation with the matrix

$$\alpha_{UV} = \mathrm{diag}(\beta_{UV},\beta_{UV},\dots,\beta_{UV}) \in GL_{nd}(\mathcal{O}_{U\cap V}).$$

The vector bundle $\pi_*P$ on $Y$ is therefore obtained by patching the free modules $\mathcal{O}_U^{nd}$ via the matrices

$$\iota_V(g_{UV})\alpha_{UV} = \alpha_{UV}\iota_U(g_{UV}) \in GL_{nd}(\mathcal{O}_{U\cap V}).$$

Hence $\det(\pi_*P)$ is obtained by patching the $\mathcal{O}_U$ via the units

$$\det(\iota_V(g_{UV}))\det(\alpha_{UV}) = \det(\alpha_{UV}).$$

Similarly, $\det(\pi_*P')$ is obtained by patching the $\mathcal{O}_U$ via the $\det(\alpha_{UV})$. It follows that $\det(\pi_*P) \cong \det(\pi_*P')$, i.e., that $\pi_*\xi = [\pi_*P] - [\pi_*P']$ has trivial determinant, i.e., that $\pi_*\xi \in SK_0(Y)$. □

Theorem 3.1 implies a simple Riemann-Roch theorem for finite maps of schemes with $\pi_*\mathcal{O}_X$ locally free. To state this result, we adapt the formalism of [FL, Ch. II]. Let $\mathcal{C}$ be the category of schemes and finite maps with $\pi_*\mathcal{O}_X$ locally free. Set

$$A(X) = K_0(X)/SK_0(X) = H^0(X,\mathbb{Z}) \oplus \mathrm{Pic}(X).$$

Since $SK_0(X)$ is an ideal, $A(X)$ is a quotient ring. If $\pi\colon X \longrightarrow Y$ is a map in $\mathcal{C}$, then $\pi_*\colon K_0(X) \longrightarrow K_0(Y)$ induces a map $\pi_*\colon A(X) \longrightarrow A(Y)$ by Theorem 3.1, and

$$\rho = (\mathrm{rank},\det)\colon K_0(X) \longrightarrow A(X)$$

yields a Riemann-Roch functor in the sense of [FL, p. 28]. By construction, the diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\rho} & H^0(X,\mathbb{Z}) \oplus \mathrm{Pic}(X) \\ \downarrow \pi_* & & \downarrow \pi_* \\ K_0(Y) & \xrightarrow{\rho} & H^0(Y,\mathbb{Z}) \oplus \mathrm{Pic}(Y) \end{array}$$

commutes for every map $\pi: X \longrightarrow Y$ in $\mathcal{C}$, which is to say:

3.4. Theorem: The Riemann-Roch Theorem holds for finite maps $\pi$ with $\pi_* \mathcal{O}_X$ locally free, relative to $(K_0, (\mathrm{rank}, \det), H^0 \oplus \mathrm{Pic})$.

## 4. $SK_0$ OF A GRADED RING

Let $A = R \oplus A_1 \oplus A_2 \oplus \ldots$ be a commutative, graded ring, and let $A_+$ denote the graded ideal $A_1 \oplus A_2 \oplus \ldots$ . If $F$ is any functor from commutative rings to abelian groups, we write $F(A,A_+)$ for the kernel of $F(A) \longrightarrow F(R)$ induced from $R \cong A/A_+$, so that $F(A) = F(R) \oplus F(A,A_+)$.

For example, it is an elementary exercise to see that all idempotents in $A$ belong to $R$, so that $H^0(A) = H^0(R)$ and $H^0(A,A_+) = 0$. From this it follows that $K_0(A,A_+) = \tilde{K}_0(A,A_+)$ and that there is a short exact sequence of abelian groups

$$(*) \qquad 0 \longrightarrow SK_0(A,A_+) \longrightarrow K_0(A,A_+) \xrightarrow{\det} \mathrm{Pic}(A,A_+) \longrightarrow 0.$$

In [Wmod], it is shown that $K_0(A,A_+)$ is naturally a continuous module over the ring $W(R)$ of Witt vectors of $R$. Here is our extension of that result.

4.1. <u>Theorem</u>: Let $A = R \oplus A_1 \oplus \dots$ be a commutative, graded ring. Then the groups $SK_0(A,A_+)$ and $Pic(A,A_+)$ are naturally continuous $W(R)$-modules in such a way that (*) is an exact sequence of $W(R)$-modules.

If $R$ contains the rational numbers, then $SK_0(A,A_+)$ and $Pic(A,A_+)$ are naturally $R$-modules, and (*) is an exact sequence of $R$-modules. (In this case, $W(R)$ is an $R$-algebra.)

<u>Proof</u>: It is enough to show that the subgroup $SK_0(A,A_+)$ of the $W(R)$-module $K_0(A,A_+)$ is closed under multiplication by $W(R)$. As pointed out in [Wmod, 1.2], it is enough to show that $SK_0(A,A_+)$ is closed under multiplication by the elements $(1 - rt^m) \in W(R)$ for all $r \in R$ and $m \geq 1$.

Fix $r \in R$ and $m \geq 1$. An additive functor $F: P(A) \longrightarrow P(A)$ was constructed in [Wmod, 1.5] such that the induced map $K_0F: K_0(A) \longrightarrow K_0(A)$ is multiplication by $m$ on the summand $K_0(R)$ and multiplication by $(1 - rt^m)$ on the summand $K_0(A,A_+)$. We need to show that $K_0F$ sends $SK_0(A,A_+)$ to itself; since

$$SK_0(A,A_+) = K_0(A,A_+) \cap SK_0(A)$$

it is enough to show that $K_0F$ sends $SK_0(A)$ to itself.

Set $S = R[s]/(s^m - r)$, and let $\sigma: A \otimes S \longrightarrow A \otimes S$ be the $S$-algebra map sendng $a_i \otimes 1$ in $A_i \otimes S$ to $a_i \otimes s^i$. If $j: A \longrightarrow A \otimes S$ denotes the natural inclusion, then the composition

$$P(A) \xrightarrow{j} P(A \otimes S) \xrightarrow{\sigma*} P(A \otimes S) \xrightarrow{j*} P(A)$$

is the functor $F$ [Wmod, 1.4]. Since $SK_0$ is natural, $j^*\sigma^* = (\sigma j)^*$

sends $SK_0(A)$ to $SK_0(A \otimes S)$. By 3.2 above, the transfer map $j_*: K_0(A \otimes S) \longrightarrow K_0(A)$ sends $SK_0(A \otimes S)$ to $SK_0(A)$. Consequently, the composition $K_1F$ sends $SK_0(A)$ to $SK_0(A)$, proving the result.

□

4.2. Remark: Multiplicaton by $(1 - rt^m)$ on $Pic(A,A_+)$ sends a rank 1 projective A-module L to $\Lambda^m(L \otimes_A P)$, where P is the A-bimodule defined in [Wmod, p. 468].

Two special cases of 4.1 are worth isolating. The first covers the case in which A is the homogeneous coordinate ring of a connected projective variety over a field.

4.3. Corollary: If k is a field and $A = k \oplus A_1 \oplus A_2 \oplus \dots$ is a commutative, graded k-algebra, then $SK_0(A)$ and $Pic(A)$ are naturally $W(k)$-modules, and

$$0 \longrightarrow SK_0(A) \longrightarrow \tilde{K}_0(A) \xrightarrow{\det} Pic(A) \longrightarrow 0$$

is a short exact sequence of $W(k)$-modules. When $char(k) = 0$, they are naturally vector spaces over k, and det is a k-linear map.

Proof: In this case $\tilde{K}_0(k) = 0$, so $\tilde{K}_0(A) = K_0(A,A_+)$. □

4.4. Corollary: If R is a commutative ring, then $NSK_0(R)$ and $NPic(R)$ are naturally $W(R)$-modules, and

$$0 \longrightarrow NSK_0(R) \longrightarrow NK_0(R) \longrightarrow NPic(R) \longrightarrow 0$$

is an exact sequence of $W(R)$-modules.

Proof: This is 4.1 when $A = R[x]$. □

4.4.1. Remark: This explains [Swan, 8.2], which points out that if $1/m \in R$ then $NPic(R)$ is a $\mathbb{Z}[1/m]$-module, while if $mA = 0$ then $NPic(R)$ is an m-torsion module. This is true of all $W(\mathbb{Z}[1/m])$-modules, resp., of all continuous $W(\mathbb{Z}/m\mathbb{Z})$-modules. The corresponding result for $NU(R)$ is a consequence of the $W(R)$-structure on $NU(R)$ given either in [WNK, 5.1] or Theorem 4.5 below.

Let us now turn to a quick study of $K_1$. If $A = R \oplus A_1 \oplus \dots$ is graded, and $nilA_+$ denotes the ideal of nilpotent elements in $A_+$, then it is well known that

$$A^* = R^* \oplus (1 + nilA_+)^*$$

$$(**) \qquad K_1(A,A_+) \cong (1 + nilA_+)^* \oplus SK_1(A,A_+).$$

(Cf. [Bass, XII.7.8].) The group $K_1(A,A_+)$ is a $W(R)$-module, and we have

4.5. Theorem: Let $A = R \oplus A_1 \oplus \dots$ be a commutative, graded ring. Then $(1 + nilA_+)^*$ and $SK_1(A,A_+)$ are $W(R)$-submodules of $K_1(A,A_+)$. In particular, (**) gives a $W(R)$-module decomposition of $K_1(A,A_+)$.

Proof: If we cite 2.2 in place of 3.2, the proof of 4.1 applies to prove that $SK_1(A,A_+)$ is a $W(R)$-submodule. To see that $(1 + nilA_+)^*$ is also a $W(R)$-submodule, we can consult the explicit formula 5.1 in [WNK]. Alternatively, if B denotes $A/nilA_+$, then

$$SK_1(A,A_+) = SK_1(B,B_+) = K_1(B,B_+)$$

[Bass, IX.1.3]. Hence the inclusion of $SK_1(A,A_+)$ in $K_1(A,A_+)$ is split by the map to $K_1(B,B_+)$. □

4.6. Theorem: Let $R \longrightarrow S$ be a map of commutative rings, and let $I$ be an ideal of $R$ mapped isomorphically onto an ideal of $S$. Then the following diagram is exact, and all maps are $W(R)$-module homomorphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & NSK_1(R) & \longrightarrow & NK_1(R) & \longrightarrow & NU(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & NSK_1(S)\oplus NSK_1(R/I) & \longrightarrow & NK_1(S)\oplus NK_1(R/I) & \longrightarrow & NU(S)\oplus NU(R/I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & NSK_1(S/I) & \longrightarrow & NK_1(S/I) & \longrightarrow & NU(S/I) & \longrightarrow & 0 \\
 & & \partial\downarrow & & \partial\downarrow & & \partial\downarrow & & \\
0 & \longrightarrow & NSK_0(R) & \longrightarrow & NK_0(R) & \longrightarrow & NPic(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & NSK_0(S)\oplus NSK_0(R/I) & \longrightarrow & NK_0(S)\oplus NK_0(R/I) & \longrightarrow & NPic(S)\oplus NPic(R/I) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & NSK_0(S/I) & \longrightarrow & NK_0(S/I) & \longrightarrow & NPic(S/I) & \longrightarrow & 0.
\end{array}$$

Proof: This is the exact diagram of abelian groups on p. 490 of [Bass]. All the groups are $W(R)$-modules and the horizontal arrows are $W(R)$-module maps by 4.4 and 4.5. Every vertical arrow except those labelled $\partial$ are $W(R)$-module maps by naturality of the module structure. It is therefore enough to show that $NK_1(S/I) \longrightarrow NK_0(R)$ is a module map. But this map is the composite of the maps

$$NK_1(S/I) \longrightarrow NK_0(S,I) \cong NK_0(R,I) \longrightarrow NK_0(R),$$

and these maps are $W(R)$-module maps by [WNK, 3.5]. □

Remark 4.6.1. If $A \longrightarrow B$ is a map of graded rings, $A = R \oplus A_1 \oplus \dots$, and $I$ is graded, then there is a similar theorem for the $W(R)$-modules $K_i(A,A_+)$, etc., which we leave to the reader.

## 5. LOCALIZATION

In this section, we study the effect of localization on $NK_0(R)$ and $NK_1(R)$. For a multiplicative set $S$ in $R$, let $[S]$ denote $\{(1-st) \in W(R)\colon s \in S\}$. This is a multiplicative set because $(1-rt)*(1-st) = (1-rst)$ in the ring $W(R)$. We shall use the following result of Vorst:

5.1. Theorem: (Vorst) If $n \leq 2$ then for every $S$:

$$NK_n(S^{-1}R) \cong [S]^{-1}NK_n(R) \cong W(S^{-1}R) \otimes_{W(R)} NK_n(R).$$

If $R$ is a $\mathbb{Q}$-algebra, so that $NK_n(R)$ is an R-module, or if $S \subseteq \mathbb{Z}$, this group also equals $S^{-1}NK_n(R)$.

Proof: See [Vorst, 1.4], [vdK, 1.6] and [WNK, 6.8]. If $M$ is any continuous $W(R)$-module, then $[S]^{-1}M$ is the same as $W(S^{-1}R) \otimes M$ by [WNK, 6.2]. □

Here is an easy application of 5.1, using 4.5 with $A = R[t]$. Consider the following diagram of $W(S^{-1}R)$-modules, whose rows are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & [S]^{-1}NSK_1(R) & \longrightarrow & [S]^{-1}NK_1(R) & \longrightarrow & [S]^{-1}NU(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow & & \\
0 & \longrightarrow & NSK_1(S^{-1}R) & \longrightarrow & NK_1(S^{-1}R) & \longrightarrow & NU(S^{-1}R) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & & & \\
 & & NSK_1(S^{-1}R_{red}) & \xrightarrow{\cong} & NK_1(S^{-1}R_{red}) & & & & \\
 & & \uparrow & & \uparrow \cong & & & & \\
 & & [S]^{-1}NSK_1(R_{red}) & \xrightarrow{\cong} & [S]^{-1}NK_1(R_{red}) & & & &
\end{array}$$

Since $NSK_1(R) \cong NK_1(R_{red})$, a diagram chase proves:

5.2. Corollary: For every multiplicative set $S$ of the ring $R$

$$NSK_1(S^{-1}R) \cong [S]^{-1}NSK_1(R) \cong W(S^{-1}R) \otimes NSK_1(R);$$
$$NU(S^{-1}R) \cong [S]^{-1}NU(R) \cong W(S^{-1}R) \otimes NU(R).$$

If $R$ is a $\mathbb{Q}$-algebra, or $S \subseteq \mathbb{Z}$, these groups also equal $S^{-1}NSK_1(R)$ and $S^{-1}NU(R)$, respectively.

Remark: The result for $S \subseteq \mathbb{Z}$ and $NU$ is classical. (See [SGA6].)

5.3. Theorem: For every multiplicative set $S$

$$NSK_0(S^{-1}R) \cong [S]^{-1}NSK_0(R) \cong W(S^{-1}R) \otimes NSK_0(R);$$
$$NPic(S^{-1}R) \cong [S]^{-1}NPic(R) \cong W(S^{-1}R) \otimes NPic(R).$$

If $R$ is a $\mathbb{Q}$-algebra, or $S \subseteq \mathbb{Z}$, these groups also equal $S^{-1}NSK_0(R)$ and $S^{-1}NPic(R)$, respectively.

Remark: The case $S \subseteq \mathbb{Z}$ was proven in [Swan, 8.1]. Theorem 5.3 supplies the answer to Swan's problem of formulating that result in greater generality.

Proof: We shall follow Swan's proof in op. cit. We can assume that $R$ is reduced as $K_0(R) \cong K_0(R_{red})$, etc. Since all functors under consideration commute with filtered colimits of rings, we may assume $R$ is a finitely generated $\mathbb{Z}$-algebra, and hence that the normalization $\bar{R}$ of $R$ is finite over $R$. Let $I$ be the conductor ideal from $\bar{R}$ to $R$; since $\bar{R}$ is finite, $I$ lies in no minimal prime of $R$ or $\bar{R}$, so that $R/I$ and $\bar{R}/I$ have lower Krull dimension. We wish to consider the K-theory exact sequences resulting from the conductor square

$$\begin{array}{ccc} R & \longrightarrow & \overline{R} \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & \overline{R}/I, \end{array}$$

and from its localization at S. Since $\overline{R}$ and $S^{-1}\overline{R}$ are reduced and normal, $NPic(\overline{R}) = NPic(S^{-1}\overline{R}) = 0$ and $NU(\overline{R}) = NU(S^{-1}\overline{R}) = 0$. Localizing the right-most exact column of W(R)-modules in 4.6 at [S], we have the map of exact column sequences of $[S]^{-1}W(R)$-modules:

$$\begin{array}{ccc} [S]^{-1}NU(R/I) & \xrightarrow{\cong} & NU(S^{-1}R/I) \\ \downarrow & & \downarrow \\ [S]^{-1}NU(\overline{R}/I) & \xrightarrow{\cong} & NU(S^{-1}\overline{R}/I) \\ \downarrow & & \downarrow \\ [S]^{-1}NPic(R) & \longrightarrow & NPic(S^{-1}R) \\ \downarrow & & \downarrow \\ [S]^{-1}NPic(R/I) & \rightarrow & NPic(S^{-1}R/I) \\ \downarrow & & \downarrow \\ [S]^{-1}NPic(\overline{R}/I) & \rightarrow & NPic(S^{-1}\overline{R}/I). \end{array}$$

The top two isomorphisms are from 5.2. Inductively, we may assume Theorem 5.3 proven for all finitely generated $\mathbb{Z}$-algebras of lower Krull dimension than R (the result being trivial if $\dim(R) = 0$). Thus the bottom two horizontal arrows are isomorphisms by induction. The 5-lemma now proves that $[S]^{-1}NPic(R) \cong NPic(S^{-1}R)$. The result for $NSK_0$ follows from the exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [S]^{-1}NSK_0(R) & \longrightarrow & [S]^{-1}NK_0(R) & \longrightarrow & [S]^{-1}NPic(R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & NSK_0(S^{-1}R) & \longrightarrow & NK_0(S^{-1}R) & \longrightarrow & NPic(S^{-1}R) & \longrightarrow & 0. \end{array}$$

□

## 6. THE FULTON-LANG FILTRATION ON $K_0$

In this section, we extend the results of the preceding sections to the subgroups $FL^nK_0(A)$ of $K_0(A)$ defined by Fulton and Lang in [FL, V.3] for commutative rings.

If A is a commutative Noetherian ring, $FL^nK_0(A)$ is defined to be the set of all $\omega \in K_0(A)$ such that for every finite family $\{Z_j\}$ of closed subsets of Spec(A) there is a bounded complex of finite projective A-modules

$$P^{\bullet}: \quad 0 \longrightarrow P^a \longrightarrow P^{a+1} \longrightarrow \dots \longrightarrow P^b \longrightarrow 0$$

such that $\omega = \sum (-1)^i [P^i]$ in $K_0(A)$, and

$$\text{codim}(Z_j \cap \text{supp}(H^i(P)), Z_j) \geq n \quad \text{for all } i \text{ and } j.$$

From [FL, V.3] we see that the $FL^nK_0(A)$ are functorial in A, $FL^1K_0(A) = \tilde{K}_0(A)$ and $FL^2K_0(A) = SK_0(A)$. Therefore, we can define $FL^nK_0(A)$ for any commutative ring A to be the direct limit of the $FL^nK_0(A_\alpha)$ over all noetherian subrings $A_\alpha$ of A.

6.1 <u>Theorem</u>: If B is a commutative A-algebra which is a finitely generated projective A-module, then the transfer map $K_0(B) \longrightarrow K_0(A)$ sends $FL^nK_0(B)$ to $FL^nK_0(A)$ for all n.

<u>Proof</u>: The usual direct limit argument shows that we may assume A and B noetherian. Suppose given an element $\omega \in FL^nK_0(B)$ and a finite family $\{Z_j\}$ of closed subsets of Spec(A). Then $\{\pi^{-1}Z_j\}$ is a family of closed subsets of Spec(B). Choose a bounded complex $P^{\bullet}$ of finite projective B-modules such that $\omega = \sum (-1)^i [P^i]$ and for all

i and j

$$\operatorname{codim}(\pi^{-1}Z_j \cap \operatorname{supp}_B(H^i(P^\bullet))) \geq n.$$

If $P^i_A$ denotes $P^i$, regarded as a finite projective A-module, then $\pi_*(\omega) = \sum (-1)^i \pi_*[P^i] = \sum (-1)^i [P^i_A]$. By the Going Down theorem,

$$\operatorname{codim}(Z_j \cap \operatorname{supp}_A H^i(P_A), Z_j) = \operatorname{codim}(\pi^{-1}Z_j \cap \operatorname{supp}_B H^i(P)) \geq n.$$

Hence $\pi_*(\omega) \in FL^nK_0(A)$ as desired. □

Fulton and Lang also define $FL^nK_0(X)$ for any noetherian scheme with an ample line bundle, in particular for quasiprojective varieties. The same proof yields the following result.

6.2 <u>Theorem</u>: Let $\pi: X \longrightarrow Y$ be a finite map of noetherian schemes with ample line bundles. Suppose that $\pi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module (ie. that $\pi$ is flat). Then the transfer map $\pi_*: K_0(X) \longrightarrow K_0(Y)$ sends $FL^nK_0(X)$ to $FL^nK_0(Y)$ for all n.

6.2.1 <u>Remark</u>: This provides another proof of Theorem 3.1 for noetherian schemes with ample line bundles.

Now we turn to module structures. The proof of Theorem 4.1 actually proves the following result:

6.3 <u>Theorem</u>: Let S(A) be any subgroup of $K_*(A)$, defined for all commutative rings A, such that the conditions

(i) any commutative ring map $A \longrightarrow B$ sends S(A) to S(B)

(ii) if B is a commutative A-algebra which is a finite projective A-module, then the transfer map $K_*(B) \longrightarrow K_*(A)$ sends S(B) to S(A) both hold. Then, for any commutative, graded ring $A = R \oplus A_1 \oplus \dots$ the group $S(A,A_+)$ is a continuous $W(R)$-submodule of $K_*(A,A_+)$.

6.4 Corollary: Let $A = R \oplus A_1 \oplus \dots$ be a commutative graded ring. Then the groups $FL^nK_0(A,A_+)$ are naturally continuous $W(R)$-submodules of $K_0(A,A_+)$. In particular, if $R$ contains the rational numbers, then $FL^nK_0(A,A_+)$ is naturally an $R$-module.

We leave the analogues of 4.3 and 4.4 to the reader, as well as the analogue of 5.3 for noetherian $R$ (the proof of 5.3 does not apply, but arguments of [Vorst] do).

## APPENDIX. PATCHING VECTOR BUNDLES

It is well known that a rank $n$ vector bundle $P$ on a scheme $X$ may be obtained by patching free $\mathcal{O}_U$-modules on some open cover $\mathcal{U} = \{U\}$ of $X$ via matrices $g_{UV} \in GL_n(U \cap V)$ on each $U \cap V$. If $P$ is given by this data, the determinant line bundle $\Lambda^nP$ is formed by patching the $\mathcal{O}_U$ together via the units $\det(g_{UV})$, so if each $g_{UV}$ belongs to $SL_n(U \cap V)$, the line bundle $\Lambda^nP$ is trivial. In this appendix we establish the well-known converse (for which we could locate no literature reference) that if $\Lambda^nP$ is trivial then, after a possible refinement of $\mathcal{U}$, we can obtain $P$ by patching via matrices in $SL_n$.

It is convenient to rephrase the above ideas in terms of nonabelian Čech cohomology. If $G$ is a sheaf of groups on $X$ such as $GL_n$ or $SL_n$, a 1-cocycle for a cover $\mathcal{U}$ with values in $G$ is a family of elements $g_{UV} \in GL_n(U \cap V)$ which are compatible on all triple intersections $U \cap V \cap W$. The cohomology set $\check{H}^1(\mathcal{U},G)$ is the quotient of the 1-cocyles by a suitable equivalence relation, and the cohomology set $\check{H}^1(X,G)$ is the direct limit of the $\check{H}^1(\mathcal{U},G)$ over all coverings $\mathcal{U}$ of $X$. We remark that each $\check{H}^1(\mathcal{U},G)$ is a subset of $\check{H}^1(X,G)$. (For more details, see [Hirz, I.3.1], [Milne, p.122], [Gir, III.3.6].)

For example, the patching data described above for the vector bundle $P$ forms a 1-cocyle for $\mathcal{U}$ with values in $GL_n$. This gives a 1-1 correspondence between the set $\check{H}^1(X,GL_n)$ and the set $P_n(X)$ of rank $n$ vector bundles on $X$. The subset $\check{H}^1(\mathcal{U},GL_n)$ corresponds to those vector bundles in $P_n(X)$ which are trivial on each of the $U$ in $\mathcal{U}$. (See [Weil], [Hirz, I.3.2.b], [Milne, p.134],.... .) The special case $n = 1$ (where $GL_1 = \mathcal{O}_X^*$) is the famous isomorphism $Pic(X) \cong \check{H}^1(X,\mathcal{O}_X^*)$.

Similarly, the set $\check{H}^1(X,SL_n)$ is in 1-1 correspondence with the set $SP_n(X)$ of "rank $n$ vector bundles on $X$ with structure group $SL_n$" [St] [Weil]. If $P \in SP_n(X)$, then the patching maps $g_{UV}$ for the underlying vector bundle belong to the $SL_n(U \cap V)$. As we observed above, this implies that $\Lambda^n(P) = \mathcal{O}_X$.

A.1. <u>Proposition</u>: Let $X$ be a scheme, and let $A = \check{H}^0(X,\mathcal{O}_X)$ denote its ring of global functions. The group $A^*$ of global units acts on the pointed set $SP_n(X)$, and the orbit set $SP_n(X)/A^*$ is isomorphic to the pointed set of all rank $n$ vector bundles $P$ on $X$ with $\det(P)$ trivial.

In particular, if $P$ is a rank $n$ vector bundle with trivial determinant, then $P$ comes from $SP_n(X)$. That is, there exists an open cover $\{U\}$ of $X$ such that $P$ may be obtained by patching free $\mathcal{O}_U$-modules together via matrices $g_{UV} \in SL_n(U \cap V)$.

Proof: For each $a \in A^*$, let $\alpha$ denote the diagonal n-by-n matrix $(a,1,\dots,1)$. If $\{g_{UV}\}$ is a 1-cocycle for $\mathcal{U}$, let ${}^a\{g_{UV}\}$ denote the cocycle $\{\alpha g_{UV}\alpha^{-1}\}$ for $\mathcal{U}$. Because $\{g_{UV}\}$ and ${}^a\{g_{UV}\}$ are equivalent 1-cocycles over $GL_n$, they define the same underlying vector bundle on $X$. It is easy to verify that this prescription gives an action of $A^*$ on each set $\check{H}^1(\mathcal{U},SL_n)$, hence on $SP_n(X)$. Now consider the short exact sequence of sheaves of groups on $X$:

$$1 \longrightarrow SL_n(\mathcal{O}_X) \longrightarrow GL_n(\mathcal{O}_X) \xrightarrow{\det} \mathcal{O}_X^* \longrightarrow 1.$$

By a diagram chase (see [Milne, III.4.5] or [Gir, III.3.3]), this gives rise to the 6-term exact sequence of pointed cohomology sets:

$$\begin{array}{ccccccccccccc} 1 \to SL_n(A) & \longrightarrow & GL_n(A) & \xrightarrow{\det} & A^* & \longrightarrow & \check{H}^1(X,SL_n) & \longrightarrow & \check{H}^1(X,GL_n) & \longrightarrow & \check{H}^1(X,\mathcal{O}_X) & \to & 1. \\ & & & & & & \| & & \| & & \| & & \\ & & & & & & SP_n(X) & \longrightarrow & P_n(X) & \xrightarrow{\Lambda^n} & \mathrm{Pic}(X) & & \end{array}$$

The description of $\Lambda^n P$ by patching makes it clear that the map $P_n(X) \longrightarrow \mathrm{Pic}(X)$ indeed sends $P$ to $\Lambda^n P$. A more careful diagram chase (see [Gir. III.3.3.3.iv]) reveals that the image of $SP_n(X)$ in $P_n(X)$ is exactly the orbit set $SP_n(X)/A^*$. □

A.2. Example: Let $X = \mathrm{spec}(A)$, where $A$ is the ring of continuous functions on the 2-sphere $S^2$ with values in $\mathbb{R}$. By [St, 18.6], $SP_2(X) \cong \pi_1(Sl_2\mathbb{R}) \cong \mathbb{Z}$. On the other hand, by [St, 18.5,26.2], the action of $-1 \in A^*$ on $SP_2(X)$ sends $n$ to $-n$ and the image of

$SP_2(X)$ is $\mathbb{N}$ in $P_2(X) \cong \mathbb{N} \sqcup \mathbb{N}$.

For our applications, we shall need a slightly stronger version of A.1. When $\pi: X \longrightarrow Y$ is a finite map, we want to describe vector bundles on $X$ using open sets of $Y$.

A.3. <u>Lemma</u>: Let $\pi: X \longrightarrow Y$ be a finite map of schemes and $P$ a rank $n$ vector bundle on $X$. Then for every point $y$ of $Y$ there is a neighborhood $U$ of $y$ such that $P$ is free on $\pi^{-1}(U)$.

<u>Proof</u>: The question being local, and $\pi$ being affine [Hart, p.128], we may asume that $Y = \text{spec}(A)$ and $X = \text{spec}(B)$, where $B$ is a finite A-module. Let $S \subset A$ be the complement of the prime $y$. Since $S^{-1}A$ is local, $S^{-1}B$ is semilocal. Since $S^{-1}P$ has constant rank, it is a free $S^{-1}B$-module [Bass, p.90]. Since $P$ is finitely generated, there is an $s \in S$ so that $P[s^{-1}]$ is a free $B[s^{-1}]$-module. □

A.4. <u>Corollary</u>: If $\pi: X \longrightarrow Y$ is a finite map, the natural map

$$\check{H}^1(Y, \pi_* GL_n) \longrightarrow \check{H}^1(X, GL_n) = P_n(X)$$

is an isomorphism.

<u>Proof</u>: If $\mathcal{U}$ is a cover of $Y$, let $\pi^{-1}(\mathcal{U})$ denote the induced cover $\{\pi^{-1}(U)\}$ of $X$. Since $(\pi_* GL_n)(U) = GL_n(\pi^{-1}(U))$, the cocycle definition of cohomology makes it clear that $\check{H}^1(\mathcal{U}, \pi_* GL_n) = \check{H}^1(\pi^{-1}(\mathcal{U}), GL_n)$. But $\check{H}^1(Y, \pi_* GL_n)$ is the union of the $\check{H}^1(\mathcal{U}, \pi_* GL_n)$, while the lemma implies that $\check{H}^1(X, GL_n)$ is the union of the $\check{H}^1(\pi^{-1}(\mathcal{U}), GL_n)$. □

A.4.1. Remark: When $n = 1$, so that $GL_1$ is commutative, A.4 follows from the Leray spectral sequence

$$H^p(Y, R^q\pi_* GL_1) \Rightarrow H^{p+q}(X, GL_1)$$

since the stalk of $R^1\pi_* GL_1$ at any point $y$ is $H^1(\mathrm{Spec}(B_y), GL_1) = \mathrm{Pic}(B_y) = 0$.

Now suppose that $P \in SP_n(X)$, i.e., that $P$ has structure group $SL_n$. We assert that there is a cover of the type $\{\pi^{-1}(U)\}$ for which $P$ may be obtained by patching via matrices in $SL_n$. To do so, note that such data determines a 1-cocycle for the cover $\{U\}$ of $Y$ with values in $\pi_* SL_n = SL_n(\pi_* \mathcal{O}_X)$. We can therefore formulate a slightly stronger result:

A.5. Proposition: Let $\pi: X \longrightarrow Y$ be a finite map. Then the natural map

$$\check{H}^1(Y, \pi_* SL_n) \longrightarrow \check{H}^1(X, SL_n) = SP_n(X)$$

is a bijection. In particular, if $P \in SP_n(X)$ then there is a cover $\{U\}$ of $Y$ such that $P$ may be obtained by patching free modules on the $\pi^{-1}(U)$ via matrices $g_{UV} \in SL_n(\pi^{-1}(U \cap V))$.

Proof: Copy the proof of A.1, using A.4. □

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