On subintegrality

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Introduction

We describe briefly some work done recently on subintegrality and its relationship with the Picard group and invertible modules. Part of the work was done jointly with L. Reid and L. G. Roberts.

All rings are assumed to be commutative with 1.

We begin by recalling from Swan [10] the notions of subintegrality and seminormalization.

An *elementary subintegral extension* is an extension $A \subseteq B$ (of rings) such that $B = A[b]$ for some b with $b^2, b^3 \in A$.

An extension $A \subseteq B$ is said to be *subintegral* if it satisfies any one of the following two equivalent conditions (see [10]):

(1) $A \subseteq B$ is integral, $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is bijective and the induced residue field extensions are trivial, i.e. $k(A \cap P) = k(P)$ for every $P \in \mathrm{Spec}\,(B)$.

(2) $A \subseteq B$ is a 'filtered union' of elementary subintegral extensions, i.e. for each $b \in B$ there exists a finite sequence $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r \subseteq B$ of subrings such that $A_i \subseteq A_{i+1}$ is elementary subintegral for each i, $0 \leq i \leq r-1$, and $b \in A_r$.

A ring A is said to be *seminormal* if it is reduced and satisfies the following condition, where K is the total quotient ring of A: $(t \in K,\ t^2, t^3 \in A) \Rightarrow t \in A$. The condition is equivalent to saying that A has no nontrivial subintegral extension contained in K.

Suppose now that A is reduced, and let K be its total quotient ring. Then the *seminormalization* of A , denoted ${}^+\!A$, is the smallest seminormal subring of K containing A. Equivalently, ${}^+\!A$ is the largest subintegral extension of A contained in K.

The following relationship between seminormality and the Picard group appears in Swan [10]:

(0.1) Theorem (Swan). *For a ring A the following three conditions are equivalent:*

(1) $\mathrm{Pic}\,(A) = \mathrm{Pic}\,(A[X_1, \ldots, X_n])$ *for some* $n \geq 1$.

(2) $\mathrm{Pic}\,(A) = \mathrm{Pic}\,(A[X_1, \ldots, X_n])$ *for all* n.

(3) A_{red} *is seminormal.*

In 1989 Dayton [1] found an interesting relationship between the Picard group and seminormalization in the case of a reduced finitely generated graded algebra over a field of characteristic zero:

(0.2) Theorem (Dayton). *Let $A = \bigoplus_{n\geq 0} A_n$ be a reduced positively graded ring such that A_0 is a field of characteristic zero and A is finitely generated as an A_0-algebra. Then there exists a functorial isomorphism of groups $\theta : \mathrm{Pic}\,(A) \xrightarrow{\sim} {}^+\!A/A$.*

The above result of Dayton was the starting point of our investigation. An obvious question is what happens if some of the hypotheses are dropped, particularly the ones on A being graded and on the characteristic being zero. We divide a discussion on this question into three cases:

Case (1) Drop the hypothesis on A being graded, but keep characteristic zero.

Case (2) Keep A graded, but allow the characteristic to be arbitrary.

Case (3) General case.

We generalize Dayton's theorem, after reformulating it suitably, to Cases (1) and (2) (Theorems (1.2), (2.1) and (2.2)). In the process of doing this, we find an elementwise characterization of subintegrality in Case (1) (Theorem (1.4)), and this leads to some interesting new examples of affine algebras (Theorems (1.6)-(1.9)). The elementwise characterization extends, with some modification, to the general case (Theorem (3.1)).

An analogue of Theorem (1.2) and Theorem (2.1) in the general case is discussed in a forthcoming paper.

We end the Introduction by observing that Dayton's theorem does not hold as stated in any of the cases listed above.

Since $\mathrm{Pic}\,(A) = 0$ for a local ring A and there exist non-seminormal local rings, the result does not hold as stated in Case (1). In a different direction, note that there exist Dedekind domains A (which are normal, hence seminormal) for which $\mathrm{Pic}\,(A) \neq 0$.

As for Case (2), we have the following

(0.3) Example. Let $A = k[t^{p+1}, t^{p+2}, \ldots, t^{2p+1}]$, where k is a field of characteristic $p > 0$ and t is an indeterminate. This is graded and satisfies all conditions of Dayton's theorem except the one on characteristic. One can show (see [9, (3.6)]) that there is an element in $\mathrm{Pic}\,(A)$ which is not killed by p. On the other hand, the group ${}^+\!A/A$ is killed by p. So $\mathrm{Pic}\,(A)$ cannot be isomorphic to ${}^+\!A/A$.

Case (1): the case of $\mathbb{Q}$*-algebras*

Consider the following example, in which A is the most elementary example of a non-seminormal ring: $A = k[t^2, t^3] \subseteq B = k[t]$ with $k = \mathbb{Q}$ and t an indeterminate. In this example A is graded and satisfies all the conditions of Dayton's theorem, and we have ${}^{+}\!A = B$. So, by Dayton's result, $\mathrm{Pic}\,(A) \cong B/A \cong k$. Localizing and completing at the origin, we get $\hat{A} = k[[t^2, t^3]] \subseteq \hat{B} = k[[t]]$ and $\hat{B}/\hat{A} \cong k$. But $\mathrm{Pic}\,(\hat{A})$ is trivial, since $\hat{A}$ is local. Now, the origin is the only singularity of A, hence the only non-seminormal point. So one would expect the isomorphism $\mathrm{Pic}\,(A) \cong B/A$ to be reflected somehow at the local level. Note that while localizing kills Pic, it introduces new units. And units and Pic are related by the following exact sequence (see [7, (2.4)]), where $\mathcal{J}(A, B)$ denotes the group of all invertible A-submodules of B:

$$1 \to A^* \to B^* \to \mathcal{J}(A, B) \to \mathrm{Pic}\,(A) \to \mathrm{Pic}\,(B).$$

In the special case when A satisfies the conditions of Dayton's theorem and $B = {}^{+}\!A$, B is also positively graded [2], $B_0 = A_0$ and $\mathrm{Pic}\,(B)$ is trivial [11]. Consequently, $A^* = A_0^* = B_0^* = B^*$ whence $\mathcal{J}(A, B) \cong \mathrm{Pic}\,(A)$.

On the other hand, in the example $\hat{A} \subseteq \hat{B}$ above, we have $(\hat{B})^*/(\hat{A})^* \cong k$ and $\mathrm{Pic}\,(\hat{A}) = 0$ whence $\mathcal{J}(\hat{A}, \hat{B}) \cong (\hat{B})^*/(\hat{A})^* \cong k \cong \hat{B}/\hat{A}$.

Thus in Dayton's situation as well as in the example $\hat{A} \subseteq \hat{B}$ above, we have $\mathcal{J}(A, B) \cong B/A$, where $B = {}^{+}\!A$. So a possible way to generalize Dayton's result would be to ask for an isomorphism $\mathcal{J}(A, {}^{+}\!A) \cong {}^{+}\!A/A$.

Our first aim is to describe such an isomorphism $\mathcal{J}(A, B) \cong B/A$ for an arbitrary subintegral extension $A \subseteq B$ of $\mathbb{Q}$-algebras.

We look first for a group homomorphism $\xi : B \to \mathcal{J}(A, B)$ such that $A \subseteq \ker(\xi)$. Now, B is additive and $\mathcal{J}(A, B)$ is multiplicative, and the addition and multiplication arise from the same ring B. The most natural way to obtain a group homomorphism in this situation seems to be via the exponential. So, roughly speaking, we should set $\xi(b) = \hat{A}e^b \cap B$, where $\hat{A}$ is a suitable completion of A. The most obvious completion is the b-adic one. However, this is not always available, for example, if b is a unit. One can get around this problem by defining $\xi(b) = I(b)|_{T=1}$, where T is an indeterminate and $I(b) = A[[T]]e^{bT} \cap B[T]$. We need here the condition that A contains $\mathbb{Q}$. The first (and, as it turns out, the main) question here is whether $\xi(b)$ is invertible as an A-submodule of B. Or, asking for a little more, whether $I(b)$ is invertible as an $A[T]$-submodule of $B[T]$. If $I(b)$ is invertible, a natural candidate for its inverse is $I(-b)$. Now, obviously $I(b)I(-b) \subseteq A[T]$ always, but in general we do not have equality. For example, if b is transcendental over A then $I(b) = 0$, as is checked easily. As it turns out, the invertibility of $I(b)$ holds under precisely the right conditions, namely we have

(1.1) Theorem. *$I(b) \in \mathcal{J}(A[T], B[T])$ if and only if the extension $A \subseteq A[b]$ is subintegral.*

Proof [7, (4.8) and (4.17)] and [8, (1.4)].

We needed just the 'if' part, and that is what we proved initially. The equivalence came as a pleasant surprise later.

Assume now that the extension $A \subseteq B$ (of $\mathbb{Q}$-algebras) is subintegral. (If A is reduced, this is equivalent to saying that $A \subseteq {}^+\!A$.) Then $A \subseteq A[b]$ is subintegral for every $b \in B$. Therefore the above theorem gives us a map $I : B \to \mathcal{J}(A[T], B[T])$. It is now an easy matter to check that this map is a homomorphism of groups with A contained in its kernel. Writing $\xi(b) = I(b)|_{T=1}$, we get a group homomorphism $\xi : B \to \mathcal{J}(A, B)$ with $A \subseteq \ker(\xi)$, and this induces a group homomorphism

$$\xi_{B/A} : B/A \to \mathcal{J}(A, B),$$

which is evidently functorial. With this notation we can state the main result in Case (1):

(1.2) Theorem. *Let $A \subseteq B$ be a subintegral extension of $\mathbb{Q}$-algebras. Then the homomorphism $\xi_{B/A} : B/A \to \mathcal{J}(A, B)$ is an isomorphism.*

Proof [7, (5.6)] and [4, (2.3)].

As noted above, we have $\mathcal{J}(A, {}^+\!A) = \mathrm{Pic}\,(A)$ under the hypothesis of (0.2). Therefore (1.2) yields Dayton's theorem as a special case:

(1.3) Corollary. Dayton's Theorem (0.2).

However, the two isomorphisms are slightly different. To be more precise, suppose the hypotheses of (0.2) are satisfied. Then we have an isomorphism

$$\xi_{{}^+\!A/A} : {}^+\!A/A \to \mathcal{J}(A, {}^+\!A) = \mathrm{Pic}\,(A)$$

given by (1.2), and also an isomorphism

$$\theta^{-1} : {}^+\!A/A \to \mathrm{Pic}\,(A)$$

given by (0.2). These two isomorphisms differ by the group automorphism of ${}^+\!A/A$ induced by the negative Euler derivation of the graded ring ${}^+\!A$, i.e. the derivation which multiplies a homogeneous element a of ${}^+\!A$ by $-\deg(a)$.

Returning to the general case of (1.2), the fact that the map $\xi_{B/A}$ is an isomorphism is proved in two stages. It is proved first for an excellent ring A of finite Krull dimension, using induction on dimension. Then the case of a general A is reduced to that of a finitely generated $\mathbb{Q}$-algebra.

The heart of the matter in this proof is Theorem (1.1), or rather its 'if' part.

Note that unlike the case of normalization, neither the definition nor the characterizations of seminormalization given above describe which elements of the total quotient ring of a reduced ring A belong to its seminormalization ${}^+\!A$. In a more general context, given an extension $A \subseteq B$ of rings, the definition and characterizations of subintegrality given above do not describe which elements b of B are such that $A \subseteq A[b]$ is subintegral. As a first step towards proving the 'if' part of Theorem (1.1) we find it necessary to develop a characterization of these elements b. Such an elementwise characterization is found over any $\mathbb{Q}$-algebra A, not necessarily reduced, and is given in the next theorem. Let us call an element b of B to be *subintegral* over A if the extension $A \subseteq A[b]$ is subintegral.

(1.4) Theorem. *Let $A \subseteq B$ be an extension of $\mathbb{Q}$-algebras, and let $b \in B$. Then b is subintegral over A if and only if there exist $c_1, \ldots, c_p \in B$ (for some $p \geq 0$) such that $b^n + \sum_{i=1}^{p} \binom{n}{i} c_i b^{n-i} \in A$ for all $n >> 0$.*
Proof [7, (4.8) and (4.17)].

We illustrate the above theorem by discussing the 'if' case with $p = 0$ (the essentially trivial case) and an example of the 'only if' case in which $p = 1$ (the first nontrivial case).

To say that an element b satisfies the criterion with $p = 0$ means that $b^n \in A$ for $n >> 0$. It is clear in this case that $A[b]$ is obtained from A by a finite sequence of elementary subintegral extensions.

As an example of the 'only if' case, consider the extension $R \subseteq S$, where $S = \mathbb{Q}[x, y, z, t]$ is the polynomial ring in four variables and $R = \mathbb{Q}[x, y, t^2, t^3, z^2 + xt, z^3 + yt]$. The extension $R \subseteq S$ is subintegral. In fact, it is the composite of two elementary subintegral extensions, namely $R \subseteq R[t]$ and $R[t] \subseteq R[t][z] = S$. One can show (see [7, (6.6)]) that $z^n + n(xt/2z)z^{n-1} \in A$ for $n \geq 8$. This means that the element $b = z$, when viewed as an element of $B = S[z^{-1}]$, satisfies the criterion of Theorem (1.4) over $A = R$ with $p = 1$ and $c_1 = xt/2z$.

Note that $R \subseteq S$ is, in fact, the generic example of a subintegral extension obtained as a composite of two elementary subintegral extensions.

In order to prove Theorems (1.1) and (1.4) we analyze first, in line with the above example, the generic situation. Namely, let $p \geq 0, N \geq 1$ be fixed integers and consider the extension

$$R = \mathbb{Q}[\gamma_n \mid n \geq N] \subseteq S = \mathbb{Q}[x_1, \ldots, x_p, z],$$

where $x_1, \ldots, x_p, z$ are indeterminates and $\gamma_n = z^n + \sum_{i=1}^{p} \binom{n}{i} x_i z^{n-i}$. Let

$$\hat{S} = \mathbb{Q}[[x_1, \ldots, x_p, z]]$$

and let $\hat{R} \subseteq \hat{S}$ be the completion of R. In this setup we prove

(1.5) Lemma. $(S \cap \hat{R}e^{z})(S \cap \hat{R}e^{-z}) = R$ *and z is subintegral over R.*
Proof Follows from [7, (3.8)].

This lemma is one of the main steps in the proof of Theorems (1.1) and (1.4). We remark that the equality $(S \cap \hat{R}e^{z})(S \cap \hat{R}e^{-z}) = R$ is similar to the equality $I(z)I(-z) = R[T]$.

Next, in order to reduce the proof of Theorem (1.2) to the case of a finitely generated $\mathbb{Q}$-algebra, we need to show that the subintegrality of an element, which appears a priori as an infinite condition in Theorem (1.4), is actually a finite condition. This is done by proving that the ring $R = \mathbb{Q}[\gamma_n \mid n \geq N]$ is a finitely generated $\mathbb{Q}$-algebra. More precisely,

(1.6) Theorem. $R = \mathbb{Q}[\gamma_n \mid N \leq n \leq 2N + 2p - 1]$.

Proof [4, (1.5)].

This completes a description of how the proof of the main result (1.2) is organized.

In the process, we have obtained the example of a ring R which is an affine algebra over $\mathbb{Q}$ in view of (1.6) and hence represents a geometric object. Note that R depends upon two integral parameters $p \geq 0, N \geq 1$, and that R is a graded subring of $\mathbb{Q}[x_1, \ldots, x_p, z]$ with weighted gradation given by $\deg z = 1$ and $\deg x_i = i$ for every i.

Investigating the structure of R, we prove first the following theorem on the relations among the γ's:

(1.7) Theorem.

(1) *Let $n \geq 2p$ be an integer, and let*

$$0 \leq d_1 < d_2 < \cdots < d_{p+1} \leq n/2$$

be any $p+1$ distinct integers. Let d be any integer with $0 \leq d \leq n/2$ and distinct from the d_i. Then there exist rational numbers $a_1, \ldots, a_{p+1}$ such that

$$\gamma_d \gamma_{n-d} = \sum_{i=1}^{p+1} a_i \gamma_{d_i} \gamma_{n-d_i}.$$

(2) *The above quadratic relations generate all relations among the γ's.*

Proof [5, (1.1) and (2.2)].

A careful analysis of the quadratic relations described above yields a computation of the Poincaré series of R:

(1.8) Theorem. *Let $P(T)$ be the Poincaré series of R, i.e.,*

$$P(T) = \sum_{n=0}^{\infty} (\dim_{\mathbb{Q}} R_n)\, T^n,$$

where R_n is the homogeneous component of R of weighted degree n. Then

$$P(T) = \frac{1 - T + T^{N+p}}{(1-T)(1-T^N)(1-T^{N+1}) \cdots (1-T^{N+p-1})}.$$

Proof [5, (2.8)].

The knowledge of the Poicaré series leads to a determination of the structure of R:

(1.9) Theorem.

(1) *R has Krull dimension $p+1$ and embedding dimension $N+2p$, and has a minimal presentation with $N+2p$ homogeneous generators and $\binom{N+p}{2}$ homogeneous relations.*

(2) *R is Cohen-Macaulay for all p, N.*

(3) *R is Gorenstein $\Leftrightarrow$ R is a complete intersection $\Leftrightarrow N+p \leq 2$.*

Proof [5, (3.2), (3.5) and (3.6)].

Case (2): the case of graded algebras over an arbitrary field

As noted in Example (0.3), Dayton's theorem does not hold in positive characteristic. The same example shows that Theorem (1.2) also fails for algebras over a field of positive characteristic. However, in the graded situation it is possible to generalize these results, after a suitable reformulation, even to the case of positive characteristic. Theorems (2.1) and (2.2) of this section describe such a generalization.

The question of extending the elementwise characterization (1.4) to the case of positive characteristic is discussed in the next section.

Let $A = \bigoplus_{n\geq 0} A_n \subseteq B = \bigoplus_{n\geq 0} B_n$ be an extension of positively graded rings with A_0 a field. Let

$$p = \begin{cases} \infty & \text{if } \operatorname{char}(A_0) = 0, \\ \operatorname{char}(A_0) & \text{if } \operatorname{char}(A_0) > 0. \end{cases}$$

Put $F_0 B = B$ and $F_i B = A + \sum_{n\geq p^i} B_n$ for $i \geq 1$. Then $F = (F_i B)_{i\geq 0}$ is a decreasing filtration on B consisting of A-subalgebras of B. Writing $F_i\mathcal{J}(A,B) = \mathcal{J}(A, F_i B)$, we get a decreasing filtration $(F_i\mathcal{J}(A,B))_{i\geq 0}$ of subgroups on $\mathcal{J}(A,B)$ with associated graded

$$\operatorname{gr}\mathcal{J}(A,B) = \bigoplus_{i\geq 0} F_i\mathcal{J}(A,B)/F_{i+1}\mathcal{J}(A,B).$$

Note that if $\operatorname{char}(A_0) = 0$ then the filtration degenerates and we get $\operatorname{gr}\mathcal{J}(A,B) = \mathcal{J}(A,B)$.

(2.1) Theorem. *Let $A \subseteq B$ be a subintegral extension of positively graded rings with A_0 a field and $B_0 = A_0$. Then there exists a natural isomorphism $\xi_{B/A} : B/A \to \operatorname{gr}\mathcal{J}(A,B)$ of groups. If $\operatorname{char}(A_0) = 0$ then the filtration degenerates, $\operatorname{gr}\mathcal{J}(A,B) = \mathcal{J}(A,B)$, and the isomorphism*

$$\xi_{B/A} : B/A \to \operatorname{gr}\mathcal{J}(A,B) = \mathcal{J}(A,B)$$

coincides with the isomorphism $\xi_{B/A} : B/A \to \mathcal{J}(A,B)$ of (1.2).

Proof [9, (4.4)].

As a special case, suppose A is reduced and finitely generated as an A_0-algebra. Then ${}^+\!A$ is positively graded and contains A as a graded subring (see [2]). Writing $F_i\mathrm{Pic}\,(A) = \ker\,(\mathrm{Pic}\,(A) \to \mathrm{Pic}\,(F_i\,{}^+\!A))$, we get on $\mathrm{Pic}\,(A)$ a decreasing filtration $(F_i\mathrm{Pic}\,(A))_{i\geq 0}$ of subgroups, with associated graded

$$\mathrm{gr}\,\mathrm{Pic}\,(A) = \bigoplus_{i\geq 0} F_i\mathrm{Pic}\,(A)/F_{i+1}\mathrm{Pic}\,(A).$$

(2.2) Theorem. *Let A be a reduced positively graded ring with A_0 a field and A finitely generated as an A_0-algebra. Then there exists a natural isomorphism $\xi_A : {}^+\!A/A \to \mathrm{gr}\,\mathrm{Pic}\,(A)$ of groups. If $\mathrm{char}\,(A_0) = 0$ then the filtration degenerates, $\mathrm{gr}\,\mathrm{Pic}\,(A) = \mathrm{Pic}\,(A)$ and the isomorphism*

$$\xi_A : {}^+\!A/A \to \mathrm{gr}\,\mathrm{Pic}\,(A) = \mathrm{Pic}\,(A)$$

differs from the isomorphism θ^{-1} of (0.2) by the group automorphism of ${}^+\!A/A$ induced by the negative Euler derivation of ${}^+\!A$.

Proof [9, (4.5)].

In the process of proving the above two theorems, we also prove the following result, which seems to be of interest in its own right:

(2.3) Theorem. *Let $A \subseteq B$ be a subintegral extension. Then for all rings C with $A \subseteq C \subseteq B$ the sequence $1 \to \mathcal{J}(A,C) \to \mathcal{J}(A,B) \to \mathcal{J}(C,B) \to 1$ of natural maps is exact.*

Proof [9, (3.3)]

Case (3): the general case

As noted in the Introduction, an analogue of Theorem (1.2) and Theorem (2.1) in the general case is discussed in a forthcoming paper. For now we describe a result (Theorem (3.1)) which extends Theorem (1.4), with a suitable modification, to the general case.

Let $A \subseteq B$ be an extension of arbitrary commutative rings. Let A'' be the set of elements b of B satisfying the criterion of Theorem (1.4), namely the following condition: There exist $c_1, \ldots, c_p \in B$ such that $b^n + \sum_{i=1}^{p} \binom{n}{i} c_i b^{n-i} \in A$ for all $n >> 0$. Theorem (1.4) can be reformulated to say that if A contains $\mathbb{Q}$ then A'' is the subintegral closure of A in B.

In order to extend this result to the general case, let us recall the notion of weak subintegrality: an extension $A \subseteq B$ of rings is said to be *weakly subintegral* if $A \subseteq B$ is integral, $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is bijective and the induced residue field extensions are purely inseparable, i.e. $k(P)$ is purely inseparable over $k(A \cap P)$ for every $P \in \mathrm{Spec}\,(B)$ (cf. condition (1) defining subintegrality in the Introduction).

(3.1) Theorem. *A'' is a subring of B and is the weak subintegral closure of A in B.*

Proof [6, (6.11)].

For an extension of $\mathbb{Q}$-algebras weak subintegrality is the same as subintegrality, and so we recover Theorem (1.4).

References

[1] B. H. Dayton, The Picard group of a reduced G-algebra, J. Pure Appl. Algebra 59 (1989) 237-253.

[2] J. V. Leahy and M. A. Vitulli, Seminormal graded rings and weakly normal projective varieties, Internat. J. Math. Sci. 8 (1985) 231-240.

[3] L. Reid and L.G. Roberts, A new criterion for weak subintegrality, Comm. Algebra 24(10) (1996) 3335-3342.

[4] L. Reid, L. G. Roberts and B. Singh, Finiteness of subintegrality, in: P. G. Goerss and J. F. Jardine (eds.), Algebraic K-Theory and Algebraic Topology, NATO ASI series, Series C, Vol. 407 (Kluwer Academic Publishers, Dordrecht, 1993) 223-227.

[5] L. Reid, L. G. Roberts and B. Singh, The structure of generic subintegrality, Proc. Indian Acad. Sci. (Math. Sci) 105 (1995) 1-22.

[6] L. Reid, L. G. Roberts and B. Singh, On weak subintegrality, J. Pure Appl. Algebra 114 (1996) 93-109.

[7] L. G. Roberts and B. Singh, Subintegrality, invertible modules and the Picard group, Compositio Math. 85 (1993) 249-279.

[8] L. G. Roberts and B. Singh, Invertible modules and generic subintegrality, J. Pure Appl. Algebra 95 (1994) 331-351.

[9] B. Singh, The Picard group and subintegrality in positive characteristic, Compositio Math. 95 (1995) 309-321.

[10] R. G. Swan, On seminormality, J. Algebra 67 (1980) 210-229.

[11] C. A. Weibel, Mayer-Vietoris sequences and module structures on NK_*, in algebraic K-theory, Evanston 1980, Lectures Notes in Mathematics 854 (Springer, Berlin, 1981) 466-493.

[12] H. Yanagihara, On an intrinsic definition of weakly normal rings, Kobe J. Math. 2 (1985) 89-98.